Math 275D Lecture 9 Notes

Daniel Raban

October 18, 2019

1 Strong Markov Property of Brownian Motion

1.1 σ-fields for stopping times

Recall, we said that a stopping time must satisfy $\{S < t\} \in \mathcal{F}_t$ for any t. We can define $S_n = \min\{\frac{k}{2^n} : \frac{k}{2^n} \geq S \geq \frac{k-1}{2^n}\}$. Then S_n is also a stopping time, and $S_n \searrow S$. Working with S_n is like working with stopping times in the discrete case. This is an example of a general technique: prove results for the discrete case and take a limit to transfer the result to the continuous case.

In the case of discrete time Markov chains, we define $\mathcal{F}_T = \{A : A \cap \{T \leq n\} \in \mathcal{F}_n \, \forall n\}$. Such events A can be expressed as $A = \bigcup_n A_n$, where $A_n = A \cap \{T = n\} \in \mathcal{F}_n$ for each n. The idea is that $\mathcal{F}_T$ is the information up to the stopping time T.

In the continuous case, we can define $\mathcal{F}_S = \{A : A \cap \{S \leq t\} \in \mathcal{F}_t \, \forall t\}$.

1.2 The strong Markov property of Brownian motion

Theorem 1.1 (strong Markov property of BM)**.** *Let $\{Y_a\}_{a \in \mathbb{R}}$ be a collection of functionals $C(\mathbb{R}) \to \mathbb{R}$, and let S be a stopping time. Then*

$$\mathbb{E}_0[Y_S \circ \theta_S \mid \mathcal{F}_S] = \mathbb{E}_{B(S)}[Y_S].$$

Remark 1.1. If we let $S = t$ be constant and set $Y_a = Y$, we get the Markov property we had before.

Example 1.1. Let $S := \inf(\{1\} \cup \{t : B_t \geq 1\})$. Let's find $\mathbb{E}[B_1 - B_S \mid \mathcal{F}_S]$. We can define $Y_S(f) = f(1 - S)$. Then $Y_S \circ \theta_S(f) = f(1)$. So

$$\mathbb{E}[B(1) \mid \mathcal{F}_S] = \mathbb{E}[Y_S \circ \theta_S \mid \mathcal{F}_S] = \mathbb{E}_{B(S)}[Y_S] = \mathbb{E}_{B(S)}[\tilde{B}(1 - S)],$$

where $\tilde{B}$ is an independent Brownian motion. If $\{S < 1\}$, then this is $\mathbb{E}_1[\tilde{B}(1 - S)] = 1$. The case for $\{S > 1\}$ will be discussed later.

Here is the idea of the proof:

Proof. We will prove that this is true for each $S_n \searrow S$. We need to show that for any $A \in \mathcal{F}_S$.

$$\mathbb{E}[\mathbb{1}_A \, \mathbb{E}_0[Y_S \circ \theta_S \mid \mathcal{F}_S]] = \mathbb{E}[\mathbb{1}_A \, \mathbb{E}_{B(S)}[\tilde{B}(1-S)]].$$

Then we only need to show that this is true for the π-system of events $\{S < t\}$ and $\{B_u \leq x\}$. Then we simplify which Ys we want to prove this for using the monotone class argument. $\square$

Remark 1.2. One subtlety to pay attention to is that $\mathcal{F}_S$ is not the same as $\mathcal{F}_S^+$; this is because S is random, so the situation is more complicated.

Example 1.2. We saw before that $\inf\{t > 0 : B_t = 0\} = 0$ a.s. If $B(0) = 1$, we will eventually hit 0 (at time $S = \inf\{t : B_t = 0\}$, say). Then we can ask the question of whether $\inf\{t > s : B_t = 0\} = 0$. The strong Markov property will let us answer this question. What should Y be?